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Topics on multi-dimensional Brox's diffusions

By

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Abstract

In this note, we give a survey of limiting behavior of multi-dimensional diffusion processes in random environments. We also report our results concerning recurrence and transience of multi-dimensional diffusion processes. In the last section, we consider the case where random environments are given by absolute values of Lévy processes.

§ 1. Introduction

Limiting behavior of random walks in random environments is well understood in the one-dimensional case. The multi-dimensional case is currently active area of research. For example, Gantert et al. [6] showed the recurrence of d independent Sinai's walks ([18]) under a uniform ellipticity condition for the environment, and pointed out that Sinai's walks are more recurrent than simple symmetric random walks. As a continuous time analogue of Sinai's walks, Brox [1] considered the one-dimensional diffusion process. Problems concerning the limiting behavior of random walks in random environments can be discussed in the framework of diffusion processes as mentioned in [26]. Recently, we obtained conditions that imply the recurrence and the transience of multi-dimensional diffusion processes in semi-stable Lévy processes ([10]). Such diffusion processes are continuous model of the random walks by Zeitouni [28] and Gantert et al. [6].

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Another direction of extensions of Brox's diffusions to multi-dimensional cases is to consider multi-parameter random environments. For this extension, we obtain conditions for the recurrence of multi-dimensional diffusion processes ([9]). This result is an extension of those by Tanaka [25] and [26].

The purpose of this note is to explain our results presented in [9] and [10] and their backgrounds. This note is constructed as follows: In Section 2, we give a brief survey concerning results of the limiting behavior of random walks and diffusion processes in random environments for one-dimensional and multi-dimensional cases. In Section 3, we explain settings of multi-dimensional diffusion processes and the results in [9] and [10]. In Section 4, we consider the case where random environments are given by absolute values of Lévy processes and show recurrence and transience of multi-dimensional diffusion processes, which do not belong to the classes of random environments studied in [9] and [10].

§ 2. Survey of limiting behavior of Sinai's walks and Brox's diffusions

§ 2.1. Sinai's walks and Brox's diffusions

We begin with a setting of one-dimensional Sinai's walks $\{X_n\}$ following [6]. Let $\xi = (\xi_x)_{x\in\mathbb{Z}}$ be a sequence of independent identically distributed random variables with values in $(0,1)$ defined on some probability space $(\Xi, \mathcal{F}, \nu)$. Let $z \in \mathbb{Z}$. For a given $\xi \in \Xi$, under P_ξ^z, we consider a Markov chain such that $P_\xi^z(X_0 = z) = 1$ and the transition probabilities are given by

$$P_\xi^z(X_{n+1} = x+1 | X_n = x) = \xi_x,$$
$$P_\xi^z(X_{n+1} = x-1 | X_n = x) = 1 - \xi_x.$$

For $x \in \mathbb{Z}$, we set $\rho_x = \rho_x(\xi) := \xi_x/(1-\xi_x)$ and denote the expectation with respect to ν by E^ν. Solomon [19] obtained the following results:

(i) If $E^\nu[\log \rho_0] = 0$, then $-\infty = \liminf_{n\to\infty} X_n < \limsup_{n\to\infty} X_n = \infty$ for almost all ξ.

(ii) If $E^\nu[\log \rho_0] > 0$, then $\lim_{n\to\infty} X_n = \infty$ for almost all ξ.

(iii) If $E^\nu[\log \rho_0] < 0$, then $\lim_{n\to\infty} X_n = -\infty$ for almost all ξ.

We remark that (i) implies $\{X_n\}$ is recurrent for almost all ξ, whereas (ii) and (iii) imply $\{X_n\}$ is transient. Following the above results, under the assumptions that $E^\nu[\log \rho_0] = 0$ and $Var(\log \rho_0) =: \sigma^2 > 0$, Sinai [18] showed that $\sigma^2(\log n)^{-2} X_n$ is convergent in distribution as $n \to \infty$. The random walks in random environments above are often called "Sinai's walks" due to the result of Sinai.

As a continuous model of Sinai's walks, Brox considered a one-dimensional diffusion process $X(t)$ described by the following formal stochastic differential equation

$$dX(t) = dB(t) - \frac{1}{2}W'(X(t))dt, \ X(0) = 0, \tag{2.1}$$

where $\{W(x), x \in \mathbb{R}\}$ is a Brownian environment independent of the Brownian motion $B(t)$; Brox obtained a similar result for $X(t)$ to that of Sinai [18]. We next explain continuous models and a derivation of a result of Sinai walks from the diffusion setting following [7] and [26].

The equation (2.1) cannot be defined as a stochastic differential equation, since the potential W is not differential in general. Hence, we treat this equation in a framework of the theory of generalized one-dimensional diffusion processes (c.f. [12]). Let $\mathcal{W}$ be the space of $\mathbb{R}$-valued functions satisfying the following:

(I) $W(0) = 0$.

(II) W is right continuous and has left limits on $[0, \infty)$.

(III) W is left continuous and has right limits on $(-\infty, 0]$.

(IV) W is a non-zero process.

We consider the Skorohod topology on $\mathcal{W}$ and regard $\mathcal{W}$ as the space of environments. Let Q be a probability measure on $\mathcal{W}$ such that $\{W(x), x \geq 0, Q\}$ and $\{W(-x), x \geq 0, Q\}$ are independent and identical in law. We call (W, Q) a random environment. Let Ω be the space of continuous paths $\omega : [0, \infty) \to \mathbb{R}$ satisfying $\omega(0) = 0$ and denote by $\omega(t)$ the value of a function $\omega(\in \Omega)$ at time t. For a fixed W, we consider a probability measure P_W on Ω such that $\{\omega(t), t \geq 0, P_W\}$ is a diffusion process with generator

$$\frac{1}{2}e^{W(x)}\frac{d}{dx}\left(e^{-W(x)}\frac{d}{dx}\right) \tag{2.2}$$

and starting at 0. A version of $X_W = \{\omega(t), t \geq 0, P_W\}$ is constructed by a standard Brownian motion with a scale change and a time change induced by W. We assume that the random environment (W, Q) and the Brownian motion are independent. We set $\mathcal{P} = P_W \times Q$. When the environment is random, we denote a diffusion process corresponding to (2.2) by $X = \{\omega(t), t \geq 0, \mathcal{P}\}$.

Brox [1] considered the case where (W, Q) is a Brownian environment and showed that the distribution of $(\log t)^{-2}X(t)$ converges as $t \to \infty$. The diffusion process corresponding to (2.2) is often called "Brox's diffusion" due to the result of Brox. The limit distribution is the same as that of the scaled Sinai's walks and its explicit form is given by Kesten [8]. Following Brox's result, Kawazu et al. [7] considered a larger class of selfsimilar random environments. A typical example is an α-stable Lévy environment, that is, $\{W(x), x \in \mathbb{R}, Q\}$ is a Lévy process with the following selfsimilarity:

$$\{W(x), x \in \mathbb{R}\} \stackrel{d}{=} \{a^{-1/\alpha}W(ax), x \in \mathbb{R}\} \quad \text{for any } a > 0, \tag{2.3}$$

where $\stackrel{d}{=}$ denotes the equality in all joint distributions under Q. For the environment satisfying $Q\{W(1) > 0\} > 0$, Kawazu et al. showed that $(\log t)^{-\alpha} X(t)$ converges as $t \to \infty$.

We remark that Sinai's walks are derived with the help of optimal sampling from Brox's diffusion and that Sinai's walks can be considered for a larger class of selfsimilar random environments ([7], [16]). Seignourel [17] considered Donsker's invariance principle for recurrent Sinai's walks with the uniform ellipticity condition for the environment. Namely, for a given $\epsilon \in (0, 1/2)$, the random environment ξ is assumed to be that $\nu(\epsilon < \xi_0 < 1 - \epsilon) = 1$. For the model, Seignourel presented discrete schemes for process in random media and obtained the convergence of Sinai's walks to Brox's diffusions.

§ 2.2. Multi-dimensional cases

The approaches of investigations in §2.1 are special for one-dimensional random walks and diffusion processes. In the multi-dimensional case, though, there have been few results.

To the knowledge of the authors, the first investigation of multi-dimensional extensions of Brox's diffusions was by Fukushima et al. [4]. They considered a diffusion process whose generator is

$$\frac{1}{2} e^{W(|x|)} \sum_{k=1}^{d} \frac{\partial}{\partial x_k} \left\{ e^{-W(|x|)} \frac{\partial}{\partial x_k} \right\}, \tag{2.4}$$

where $|x| = \sqrt{x_1^2 + \cdots + x_d^2}$ and W is a one-dimensional standard Brownian motion. The existence of the diffusion process X_W corresponding to the generator (2.4) is guaranteed by the Dirichlet form theory (c.f. [5]). Using Ichihara's recurrence test ([11], [3]), Fukushima et al. showed the recurrence of the diffusion process X_W for almost all environments in any dimension. They also considered cases where random environments are given by some centered Gaussian random fields and obtained conditions of the fields that imply the recurrence and transience of X_W for almost all environments.

Tanaka [25] considered the case where the random environment is given by Lévy's multi-dimensional Brownian motion. We denote by $\mathcal{W}_0$ the space of continuous functions on $\mathbb{R}^d$ vanishing at the origin and assume that its topology is equipped with locally uniform convergence. Let Q be the probability measure on $\mathcal{W}_0$ such that $\{W(x), x \in \mathbb{R}^d, Q\}$ is Lévy's Brownian motion with a d-dimensional time. For a fixed W, Tanaka considered a diffusion process X_W corresponding to the following generator

$$\frac{1}{2} e^{W}(x) \sum_{k=1}^{d} \frac{\partial}{\partial x_k} \left\{ e^{-W(x)} \frac{\partial}{\partial x_k} \right\}. \tag{2.5}$$

Using Ichihara's recurrence test and the ergodic property of Lévy's Brownian motion, Tanaka showed the recurrence of the diffusion processes X_W for almost all environments in any dimension. Tanaka also considered the cases where random environments are generated by the absolute values of Lévy's Brownian motion and showed that $X_{|W|}$ is also recurrent for almost all environments in any dimension; however, $X_{-|W|}$ is transient for any $d \geq 2$. In [26], Tanaka remarked that the recurrence of X_W is valid when $\{W(x)\}$ is replaced by any continuous random fields $\{V(x)\}$ in $\mathbb{R}^d$ satisfying some conditions. In §3.1, we explain our result for semi-selfsimilar environment which extends the results of [25] and [26].

Mathieu [14] considered the diffusion process whose generator is given by (2.5) and obtained results on localization for some random fields. As mentioned in [25], Tanaka's investigation above was motivated by the results in [14]. Mathieu also obtained conditions such that the distributions of scaled diffusion processes converge as $t \to \infty$. An example of random environments is the absolute value of the Lévy's Brownian motion. Following [14], Kim [13] considered the random set of trajectories of multi-dimensional diffusion processes whose generator is given by (2.5). Kim obtained a limit theorem for the shape of the full trajectory of the diffusion processes and gives two examples of random environments; one is the absolute values of the Lévy's Brownian motion, the other is a random environment consisting of d independent one-dimensional reflected non-negative Brownian motions, which is a model studied in [22].

The limiting behavior of multi-dimensional Sinai's walks is also less well understood than one-dimensional case. In [2], many open questions are presented. For example, the classification of transience and recurrence of multi-dimensional Sinai's walks is still an open question (Open question 2.23 in [2]). In Appendix of [28], Zeitouni considered a product of d one-dimensional Sinai's walks under the uniform ellipticity condition for the random environment ξ. Zeitouni showed the recurrence of the random walks for almost all environments in any dimension using a random conductance model. Recently, Gantert et al. [6] showed the recurrence of d independent Sinai's random walks in the same random environments under the uniform ellipticity condition. They used estimates of quenched return probabilities to the origin of the one-dimensional random walks in random environments. As a continuous analogues of models in [28] and [6], the limiting behavior of multi-dimensional diffusion processes generated by d independent Brox's diffusions was studied in [23]. We remark that the uniform ellipticity condition for environments is not assumed for the models. In §3.2, we explain our recent study concerning the multi-dimensional diffusion processes.

§ 3. Settings and results in [9] and [10]

§ 3.1. Multi-dimensional diffusion processes in multi-parameter fields

In this subsection, we consider cases where environments are given by multi-parameter fields. We first consider a case of non-random environments. Let W be a locally bounded and measurable function on $\mathbb{R}^d$, and let $W(0) = 0$. For a fixed W, we consider a diffusion process X_W whose generator is (2.5). Fixing $a > 1$, we let E_n be the set $\{x \in \mathbb{R}^d, |x| < a^n\}$ for $n \in \mathbb{Z}$. We denote $E_n \setminus E_{n-1}$ by D_n for $n \in \mathbb{Z}$. Then, a sufficient condition of the environments for the recurrence of X_W is given as follows:

Proposition 3.1 ([9]). *If there exists a constant $c \in \mathbb{R}$ such that*

$$\inf_{x \in D_n} W(x) \geq n(d-2)\log r - c$$

holds for infinitely many $n \in \mathbb{N}$, then X_W is recurrent in any dimension.

We next explain random environments' cases. Fixing $H > 0$, we define a mapping T from Borel measurable functions on $\mathbb{R}^d$ to themselves by

$$Tf(x) := a^{-H} f(ax). \tag{3.1}$$

We set a probability measure Q on $\mathcal{W}_0$ satisfying the law of TW being equal to that of W. Then, T is a measure preserving transformation and W has semi-selfsimilarity. We remark that this setting is an extension of Tanaka's argument in [26]. Then, for a larger class of random environments, sufficient conditions for the recurrence of X_W are given as follows:

Theorem 3.2 ([9]). *If T is weakly mixing and there exists a positive constant ϵ such that*

$$Q\left\{\inf_{x \in D_1} W(x) \geq \epsilon\right\} > 0, \tag{3.2}$$

then X_W is recurrent for almost all environments in any dimension.

When (W, Q) is a Gaussian field, the condition (3.2) is described by its correlation function. Let $\{W(x), x \in \mathbb{R}^d, Q\}$ be a family of random variables such that the $\mathbb{R}^n$-valued random variables $(W(x_1), \ldots, W(x_n))$ have an n-dimensional Gaussian distribution for any $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathbb{R}^d$. We assume that W is continuous on $\mathbb{R}^d$ almost surely, $W(0) = 0$ almost surely and that $E[W(x)] = 0$ for $x \in \mathbb{R}^d$. We set $K(x, y) := E[W(x)W(y)]$ for $x, y \in \mathbb{R}^d$. By extending Tanaka's argument in [25] to abstract Wiener spaces, the criteria for the recurrence of X_W are given as follows:

Theorem 3.3. *For a Gaussian field (W, Q), we assume the following conditions:*
(i) *There exists a positive constant ϵ such that*

$$\inf_{x \in D_1} \int_{D_1} K(x, y) dy \geq \epsilon.$$

(ii) *The law of $T^n W$ is equal to that of W for any $n \in \mathbb{Z}$ and that*

$$\lim_{n \to \infty} r^{-nH} \sup_{x, y \in D_1} K(r^n x, y) = 0.$$

Then, the diffusion process X_W is recurrent for almost all environments in any dimension.

An example satisfying the conditions above is a fractional Brownian field with the Hurst parameter $H \in (0, 1)$. Here, the fractional Brownian field is a Gaussian field such that $E[W(0)] = 0$ for $x \in \mathbb{R}^d$ and the covariance between $W(x)$ and $W(y)$ is given by

$$K(x, y) = \frac{1}{2}(|x|^{2H} + |y|^{2H} - |x - y|^{2H}), \ x, y \in \mathbb{R}^d.$$

We remark that when $H = 1/2$, it is Lévy's Brownian motion. We also remark that fractional Brownian fields have selfsimilarities, that is, (3.1) is satisfied for any $a > 0$.

§ 3.2. Random environments generated by Lévy processes

In this subsection, we explain multi-dimensional diffusion processes generated by d independent diffusion processes in semi-stable Lévy environments. We set a probability measure Q on $\mathcal{W}$ such that $\{W(x), x \geq 0, Q\}$ and $\{W(-x), x \geq 0, Q\}$ are independent, identical in law and strictly semi-stable Lévy processes with index $\alpha \in (0, 2]$. The environment has semi-selfsimilarity and (2.3) is satisfied for some $a > 0$. This a is termed an epoch. We set

$$r = \inf\{a > 1 : a \text{ satisfies } (2.3)\}. \tag{3.3}$$

In this note, we call (W, Q) an (r, α)-semi-stable Lévy environment. We remark that the trivial process, that is $W(x) = cx$ almost surely for $x \geq 0$ with $c \neq 0$, is also a non-zero, strictly semi-stable Lévy process (Remark 13.17 in [15]). If $r = 1$, (W, Q) is not only semi-selfsimilar, but selfsimilar (Theorem 13.11 in [15]). In this case, (W, Q) is an α-stable Lévy environment. If $\alpha = 2$, then $r = 1$ and (W, Q) is a Brownian environment (Theorem 14.1 in [15]). Refer to [15] for more properties of semi-stable Lévy processes.

For a fixed W, we consider a d-dimensional diffusion process starting at 0, $X_W = \{X_W^k(t), t \geq 0, k = 1, \ldots, d\}$ whose generator is

$$\sum_{k=1}^{d} \frac{1}{2} \exp\{W(x_k)\} \frac{\partial}{\partial x_k} \left\{ \exp\{-W(x_k)\} \frac{\partial}{\partial x_k} \right\}. \tag{3.4}$$

Then, the dichotomy of recurrence and transience of the multi-dimensional diffusion process is shown as follows:

Theorem 3.4 ([10]). (i) *If $\{-W(x), x \geq 0, Q\}$ is not a subordinator, then X_W is recurrent for almost all environments in any dimension.*

(ii) *If $\{-W(x), x \geq 0, Q\}$ is a subordinator, then X_W is transient for almost all environments in any dimension.*

We next consider the cases of d independent environments. Let Q_k be the probability measure on $\mathcal{W}$ satisfying the following.

(i) $\{W_k(-x_k), x_k \geq 0, Q_k\}$ is a strictly (l_k, α_k)-semi-stable or α_k-stable Lévy process.

(ii) $\{W_k(x_k), x_k \geq 0, Q_k\}$ is a strictly (r_k, β_k)-semi-stable or β_k-stable Lévy process.

(iii) They are independent.

We define an environment $(\mathbf{W}, \mathbf{Q})$ by $\{(W_k, Q_k), k = 1, \ldots, d\}$ with each (W_k, Q_k) being independent. We remark that Suzuki [20] studied the case where $d = 1$ with independent an α-stable and a β-stable Lévy environments, and obtained some convergence theorems. For a fixed $\mathbf{W}$, we consider a d-dimensional diffusion process starting at 0, $X_{\mathbf{W}} = \{X^{(k)}_{W_k}(t), t \geq 0, k = 1, \ldots, d\}$ whose generator is

$$\sum_{k=1}^{d} \frac{1}{2} \exp\{W_k(x_k)\} \frac{\partial}{\partial x_k} \left\{ \exp\{-W_k(x_k)\} \frac{\partial}{\partial x_k} \right\}. \tag{3.5}$$

For the multi-dimensional diffusion process, the dichotomy is also given as follows:

Theorem 3.5 ([10]). (i) *If neither $\{-W_k(-x_k), x_k \geq 0, Q_k\}$ nor $\{-W_k(x_k), x_k \geq 0, Q_k\}$ is a subordinator for any k, then $X_{\mathbf{W}}$ is recurrent for almost all environments in any dimension.*

(ii) *If either $\{-W_k(-x_k), x_k \geq 0, Q_k\}$ or $\{-W_k(x_k), x_k \geq 0, Q_k\}$ is a subordinator for some k, then $X_{\mathbf{W}}$ is transient for almost all environments in any dimension.*

We remark that diffusion processes in Theorem 3.4 and 3.5 are continuous models of random walks in [6] and [28], respectively. The random environments in [6] and [28] correspond to Brownian environments for our models.

§ 4. Other models

In this section, we give simple examples that do not belong to the class of random environments mentioned in Section 3. This is, because, to explain how to use Ichihara's recurrence and transience tests. In addition, we explain novelties of the environments.

§ 4.1. Random environments generated by the absolute value of a semi-stable Lévy process

Tanaka [24] studied the limiting behavior of one-dimensional diffusion processes in non-negative and non-positive reflected Brownian motions. Tanaka showed that the scaling properties are the same as that of Brox-type diffusion and obtained the probabilistic interpretations of the limit distributions for both environments. Instead of the reflected Brownian motions, we consider the cases where the environments are given by absolute values of non-zero strictly (r, α)-semi-stable Lévy processes. In the case where $\alpha \in (1, 2]$, the semi-stable Lévy process is point recurrent (Remark 43.12 in [15]) and the environment $(|W|, Q)$ contains two valleys of $|W|$ connected at the origin with depth r almost surely. This and semi-selfsimilarity of $|W|$ imply that the distribution of $(\log t)^{-\alpha} X_{-|W|}(t)$ converges weakly to a functional of $|W|$ along a subsequence of t ([21]).

In [22], the limiting behavior of multi-dimensional diffusion processes in random environments generated by independent non-positive or non-negative reflected Brownian motions is concerned. In this subsection, we consider the case where random environments are generated by absolute values of strictly semi-stable Lévy processes which is an extension of the results in [22]. We remark that if the environment (W, Q) satisfies that $Q\{W(1) > 0\} = 0$, then $(|W|, Q)$ is a subordinator hence a Lévy process; otherwise $(|W|, Q)$ is not a Lévy process. For the environments, we obtain the following results.

Theorem 4.1. *Let W be a non-zero strictly semi-stable Lévy process.*

(i) *Suppose a multi-dimensional diffusion process $X_{|W|}$ whose generator is given by*

$$\sum_{k=1}^{d} \frac{1}{2} \exp\{|W(x_k)|\} \frac{\partial}{\partial x_k} \left\{ \exp\{-|W(x_k)|\} \frac{\partial}{\partial x_k} \right\}. \tag{4.1}$$

Then, for almost all environments $X_{|W|}$ is recurrent for any $d \geq 2$.

(ii) *Suppose a multi-dimensional diffusion process $X_{-|W|}$ whose generator is given by*

$$\sum_{k=1}^{d} \frac{1}{2} \exp\{-|W(x_k)|\} \frac{\partial}{\partial x_k} \left\{ \exp\{|W(x_k)|\} \frac{\partial}{\partial x_k} \right\}. \tag{4.2}$$

Then, for almost all environments $X_{-|W|}$ is transient for any $d \geq 2$.

Outline of the proof.

To show (i) of the theorem, we use Ichihara's recurrence test ([11], [3]). It is sufficient to show that for almost all environments

$$\int_1^\infty s^{1-d} \left\{ \int_{S^{d-1}} \exp\left\{ -\sum_{j=1}^{d} |W(s\sigma_j)| \right\} d\sigma \right\}^{-1} ds = \infty, \tag{4.3}$$

where $d\sigma$ is the normalized uniform measure on S^{d-1}.

If $Q\{W(1) > 0\} > 0$, then $Q\{|W(x)| \geq W(x) \text{ for any } x \in \mathbb{R}\} = 1$ and for almost all environments

$$\int_{S^{d-1}} \exp\left\{-\sum_{j=1}^{d} |W(s\sigma_j)|\right\} d\sigma \leq \int_{S^{d-1}} \exp\left\{-\sum_{j=1}^{d} W(s\sigma_j)\right\} d\sigma$$

for any $s \in \mathbb{R}$. Hence we obtain that

$$\begin{aligned} &(\text{the left-hand side of (4.3)}) \\ &\geq \int_1^{\infty} s^{1-d} \left\{\int_{S^{d-1}} \exp\left\{-\sum_{j=1}^{d} W(s\sigma_j)\right\} d\sigma\right\}^{-1} ds \end{aligned} \tag{4.4}$$

for almost all environments. In the proof of (i) of Theorem 3.4, the infiniteness of the right-hand side of (4.4) is shown by using a property of strictly semi-stable Lévy processes and their ergodicity. In the case $Q\{W(1) > 0\} = 0$, (4.3) is shown by the proof of Theorem 3.4.

To show (ii) of the theorem, we apply Ichihara's transience test ([11]) to the generator

$$\sum_{k=1}^{d} \frac{\partial}{\partial x_k}\left\{\exp\left\{\sum_{j=1}^{d} |W(x_j)|\right\} \frac{\partial}{\partial x_k}\right\}$$

and shall show that for any $d \geq 2$

$$\int_1^{\infty} s^{1-d} \exp\left\{-\sum_{k=1}^{d} |W(s\sigma)|\right\} ds < \infty \tag{4.5}$$

for σ which belongs to some subset of S^{d-1} with a positive surface measure for almost all environments. Then, it is sufficient to consider the case where $d = 2$.

First, we show that for a fixed $\epsilon \in (0,1)$ there exists $I \subset [\epsilon, 1]$ such that $|I| > 0$ and for any $u \in I$

$$E\left[\int_1^{\infty} s^{-1} \exp\{-|W(su)|\}\, ds\right] < \infty. \tag{4.6}$$

To show (4.6), we use the following property of Lévy processes obtained by Tanaka [27]:

Proposition 4.2. *If $\{W(x), x \geq 0, Q\}$ is a non-zero Lévy process, then for any $\xi > 0$*

$$E[e^{-\xi|W(x)|}] \leq Cx^{-1/4} \quad \textit{for any } x \geq 0, \tag{4.7}$$

where C depends on ξ but not on x.

We thus obtain that

$$
\begin{aligned}
&\text{(the left-hand side of (4.6))}\\
&= \int_1^\infty s^{-1} E[\exp\{-|W(su)|\}]ds\\
&= 4Cu^{-1/4} \ \le 4C\epsilon^{-1/4} \qquad (4.8)
\end{aligned}
$$

for any $u \in [\epsilon, 1)$, where C is the constant in (4.7) when $\xi = 1$.

Secondly, we show that the exceptional set of probability zero does not depend on u. The inequality (4.8) and Fubini's theorem imply that

$$
\begin{aligned}
&\int_\epsilon^1 E\left[\int_1^\infty s^{-1}\exp\{-|W(su)|\}ds\right] du\\
&= E\left[\int_\epsilon^1 \left(\int_1^\infty s^{-1}\exp\{-|W(su)|\}ds\right) du\right] \ \le 4C\frac{(1-\epsilon)}{\epsilon^{1/4}}
\end{aligned}
$$

and that almost surely

$$
\int_\epsilon^1 \left(\int_1^\infty s^{-1}\exp\{-|W(su)|\}ds\right) du < \infty.
$$

Hence we obtain that for a fixed $\epsilon \in (0,1)$ there exists $I \subset [\epsilon, 1]$ such that $|I| > 0$ and

$$
\int_1^\infty s^{-1}\exp\{-|W(su)|\}ds < \infty, \quad u \in I \tag{4.9}
$$

for almost all environments. □

§ 4.2. A two-dimensional diffusion process

Gantert et al. [6] considered a two-dimensional random walk $\{X_n, Y_n, n \in \mathbb{N}\}$ such that $\{X_n\}$ is a random walk in the random environment studied by Sinai [18], $\{Y_n\}$ is a centered random walk that converges weakly to a strictly α-stable Lévy process with $\alpha \in (1, 2]$ under a suitable scaling, and they are independent. They showed the recurrence of the random walk for almost all environments.

For a continuous model, a two-dimensional diffusion process formed by independent two processes where one is a one-dimensional Brownian motion and the other is a one-diffusion process in a non-positive reflected Brownian environment studied by Tanaka [24] is considered in [10]. For the diffusion process, the transient property is shown. In this note, we extend the model to the cases where the random environments are given by the negative absolute values of Lévy processes. The generator of the two-dimensional diffusion process is given by

$$
\sum_{k=1}^2 \frac{\partial}{\partial x_k}\left\{\exp\{|W(x_2)|\}\frac{\partial}{\partial x_k}\right\}. \tag{4.10}
$$

Then, as a corollary of (ii) of Theorem 4.1, we obtain the following transient property:

Corollary 4.3. *Let W be a non-zero Lévy process. If we assume that $\{B^1(t)\}$ and $\{X^2_{-|W|}(t)\}$ are independent, then $\{B^1(t), X^2_{-|W|}(t)\}$ is transient for almost all environments.*

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